Computing the distance distribution of systematic non-linear codes

Eleonora Guerrini (guerrini@posso.dm.unipi.it)
Department of Mathematics, University of Trento, Italy.

Emmanuela Orsini (orsini@posso.dm.unipi.it)
Department of Mathematics, University of Pisa, Italy.

Massimiliano Sala (msala@bcri.ucc.ie)
Boole Centre for Research in Informatics, UCC Cork, Ireland,
Department of Mathematics, University of Trento, Italy.

Abstract

The most important families of non-linear codes are systematic. A brute-force check is the only known method to compute their weight distribution and distance distribution. On the other hand, it outputs also all closest word pairs in the code. In the black-box complexity model, the check is optimal among closest-pair algorithms. In this paper we provide a Gröbner basis technique to compute the weight/distance distribution of any systematic non-linear code. Also our technique outputs all closest pairs. Unlike the check, our method can be extended to work on code families.

Keywords: Gröbner basis, distance distribution, Hamming distance, non-linear code.

1 Introduction

In the celebrated paper [Sha48] by Shannon, the mathematical foundation of coding theory was laid. Codes presented in that paper are non-linear, notably including codes used in the proof of the landmark Capacity theorem. Although no proof for linear codes of the Capacity theorem was known until the 60's ([Gal63]), coding theorists have been studying only linear codes, with a few exceptions ([Pre68],[BvLW83]). This is not surprising, since linear codes have a nice structure, easy to study and leading to efficient implementations. Still, it is well-known that some non-linear codes have a higher distance (or a better distance distribution) that any linear code with the same parameters ([Pre68], [PHB98]). This translates into a superior decoding performance.

A class of non-linear codes that has received some attention is composed of systematic non-linear codes, since they are easier to encode. Moreover, the best known non-linear codes are systematic or equivalent to systematic ones ([BvLW83],[PHB98], [HKC+94]), so that no performance degradation is shown while restricting to systematic codes. In this paper our main result is a method, based on Gröbner basis computations, that allows to determine the (distance and) distance distribution of a systematic code. No other method is known, except for the "brute-force" approach consisting of checking the mutual distance of any pair of codewords. Both our method and the brute-force approach output the *closest pairs*, i.e. the codeword pairs whose distance is minimal. We show that the complexity of the "brute-force" approach matches the complexity of the closest pair problem and so this method is optimal (the proof is given within the black-box complexity model with distance oracle).

2 Notation and preliminary results

Let $m \geq 1$ be a natural number. Let $\mathbb{K}$ be a field, $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$ and I be an ideal in the polynomial ring $\mathbb{K}[Y] = \mathbb{K}[y_1, \ldots, y_m]$. For any $q \geq 1$, we denote by $E_q[Y] \subset \mathbb{K}[Y]$ the set of polynomials $E_q[Y] = \{y_1^q - y_1, \ldots, y_m^q - y_m\}$. Given a polynomial $f \in \mathbb{K}[Y]$, we denote by $\mathcal{V}(f)$ the set of all zeros of f in $(\overline{\mathbb{K}})^m$. Given an ideal $I \subseteq \mathbb{K}[Y]$, we denote by $\mathcal{V}(I)$ the set of all zeros of I in $(\overline{\mathbb{K}})^m$. Let $S \subset (\overline{\mathbb{K}})^m$. The set of all polynomials $f \in \mathbb{K}[Y]$ such that $f(a_1, \ldots, a_m) = 0$ for any point $(a_1, \ldots, a_m)$ in S forms an ideal in polynomial ring $\mathbb{K}[Y]$, called the **vanishing ideal** of S and denoted by $\mathcal{I}(S)$. If $L \subset \mathbb{K}[Y]$, we denote by $\langle L \rangle$ the ideal in $\mathbb{K}[Y]$ generated by L.

Let $\mathbb{F}_q$ be the finite field with q elements and $(\mathbb{F}_q)^m$ be the natural m–dimensional vector space over $\mathbb{F}_q$.

Definition 2.1. *Let $1 \leq t \leq m$. We denote by $\mathcal{M}_{m,t,q}$ the following set:*

$$\mathcal{M}_{m,t,q} = \{y_{h_1} \cdots y_{h_t} \mid 1 \leq h_1 < \ldots < h_t \leq m\}.$$

From now on, we will use $\mathcal{M}_{m,t}$ instead of $\mathcal{M}_{m,t,q}$. We will also shorten $y_{h_1} \cdots y_{h_t}$ to Y_L, where $L = \{h_1, \ldots, h_t\}$.

Let s be an integer $1 \leq s \leq m-1$. We fix in $\mathbb{F}_q[y_1, \ldots, y_s, t_1, \ldots, t_{m-s}] = \mathbb{F}_q[Y, T]$, the lexicographic order $y_1 < y_2 < \ldots < y_s < t_1 < \ldots < t_{m-s}$. Let I be an ideal in $\mathbb{F}_q[Y, T]$ we denote by $G(I) \subset \mathbb{F}_q[Y, T]$ the minimal reduced Gröbner basis of I w.r.t. $<$ ordering ([Buc65,Buc06,CLO92]).

Let $\phi : (\mathbb{F}_q)^k \to (\mathbb{F}_q)^n$ be an injective function and let C be $\mathrm{Im}(\phi)$. We say that C is an (n, k, q) **code**. Any $c \in C$ is called a **word**. Let $\pi : (\mathbb{F}_q)^n \to (\mathbb{F}_q)^k$ be $\pi(a_1, \ldots, a_n) = (a_1, \ldots, a_k)$. We say that C is **systematic** if $(\pi \circ \phi)(v) = v$ for any $v \in (\mathbb{F}_q)^k$. We denote by $\mathcal{C}(n, k, q)$ the class of systematic (n, k, q) codes.

For any two vectors $v_1, v_2 \in (\mathbb{F}_q)^n$, $\mathrm{d}(v_1, v_2)$ denotes the **(Hamming) distance** between v_1 and v_2. For any $v \in (\mathbb{F}_q)^n$, $\mathrm{w}(v)$ denotes the **weight** of v. Let $C \in \mathcal{C}(n, k, q)$, $\mathrm{d}(C)$ denotes the distance of C, $B_i = B_i(C)$ denotes

the number of codewords in C with weight i. Integer set $\{B_0, B_1, \ldots, B_n\}$ is called the **weight distribution** of C and $A_i = A_i(C)$ denotes the number of (unordered) codeword pairs with distance i. Integer set $\{A_1, \ldots, A_n\}$ is called the **distance distribution** of C.

A code C is **distance-invariant** if for any $1 \leq i \leq n$ and any $c, c' \in C$,

$$|\{y \in C \mid \mathrm{d}(c, y) = i\}| \;=\; |\{y \in C \mid \mathrm{d}(c', y) = i\}| \,.$$

Clearly linear codes are distance-invariant. The distance distribution of distance-invariant codes (containing the zero vector) can be immediately obtained from their weight distribution. Note that many optimal codes are distance-invariant codes ([HKC$^+$94]).

We want to show some relations between some sets of vectors in $(\mathbb{F}_q)^n$ having a weight property and the Gröbner bases of associated ideals. We use polynomial ring $\mathbb{F}_q[Y]$ with any term-order, since our results here hold for any (admissible) term-order.
The *elementary symmetric functions* are the polynomials $\sigma_1 = y_1 + \cdots + y_m, \ldots, \sigma_m = y_1 y_2 y_3 \cdots y_{m-2} y_{m-1} y_m$.

Definition 2.2. *Let $t \in \mathbb{N}$ be s.t. $1 \leq t \leq m$. We denote by $I_{m,t}$ the ideal:*

$$I_{m,t} = \langle \{\sigma_t, \ldots, \sigma_m\} \cup E_q[Y] \rangle \subset \mathbb{F}_q[Y] \,.$$

Our aim is to determine the reduced Gröbner basis of $I_{m,t}$. For this, we need a preliminary result.

Lemma 2.3. *Let $l \in \mathbb{N}$ be such that $1 \leq l \leq m-1$. Then*

$$\mathcal{M}_{m,l} \subset \langle \mathcal{M}_{m,l+1} \cup \{\sigma_l\} \cup E_q[Y] \rangle \,.$$

Proof. Let I be the ideal $I = \langle \mathcal{M}_{m,l+1} \cup \{\sigma_l\} \cup E_q[Y] \rangle$.
Let $\mathcal{A} = \{A \subset \{1, .., m\} \mid |A| = l\}$ and $L = \{i_1, \ldots, i_l\} \in \mathcal{A}$. We have:

$$\sigma_l = \sum_{\substack{A \in \mathcal{A} \\ \{i_1\} \subset A}} Y_A + \sum_{\substack{B \in \mathcal{A} \\ \{i_1\} \not\subset B}} Y_B \;\; y_{i_1}^{q-1} \sigma_l = y_{i_1}^{q-1} \sum_{\substack{A \in \mathcal{A} \\ \{i_1\} \subset A}} Y_A + y_{i_1}^{q-1} \sum_{\substack{B \in \mathcal{A} \\ \{i_1\} \not\subset B}} Y_B \,. \qquad (1)$$

We note that $y_{i_1} \sum_{\{i_1\} \not\subset B} Y_B$ is a sum of monomials in $\mathcal{M}_{m,l+1}$, hence $y_{i_1}^{q-2}(y_{i_1} \sum_{\{i_1\} \not\subset B} Y_B) = y_{i_1}^{q-1} \sum_{\{i_1\} \not\subset B} Y_B$ is in I. By (1), since $y_{i_1}^{q-1} \sigma_l \in I$, we obtain that

$$y_{i_1}^{q-1} \sum_{\substack{A \in \mathcal{A} \\ \{i_1\} \subset A}} Y_A \;\in\; I \,.$$

From (1), since any monomial in $\sum_{\{i_1\} \subset A} Y_A$ contains y_{i_1}, by reduction w.r.t. $E_q[Y]$ we have $\sum_{\substack{A \in \mathcal{A} \\ \{i_1\} \subset A}} Y_A \;\in\; I$. We can write:

$$\sum_{\substack{A \in \mathcal{A} \\ \{i_1\} \subset A}} Y_A = \sum_{\substack{B \in \mathcal{A} \\ \{i_1, i_2\} \subset B}} Y_B + \sum_{\substack{C \in \mathcal{A} \\ \{i_1, i_2\} \not\subset C, \{i_1\} \subset C}} Y_C \,, \qquad (2)$$

and from (2) we get

$$y_{i_2}^{q-1} \sum_{\substack{A\in\mathcal{A}\\ \{i_1\}\subset A}} Y_A = y_{i_2}^{q-1} \sum_{\substack{B\in\mathcal{A}\\ \{i_1,i_2\}\subset B}} Y_B + y_{i_2}^{q-1} \sum_{\substack{\{i_1,i_2\}\not\subset C\\ \{i_1\}\subset C, C\in\mathcal{A}}} Y_C \,.$$

Since $y_{i_2}^{q-1} \sum_{\substack{\{i_1,i_2\}\not\subset B\\ \{i_1\}\subset B}} Y_B = y_{i_2}^{q-2}(y_{i_2} \sum_{\substack{\{i_1,i_2\}\not\subset B\\ \{i_1\}\subset B}} Y_B)$ and $y_{i_2} \sum_{\substack{\{i_1,i_2\}\not\subset B\\ \{i_1\}\subset B}} Y_B$ is in I, then the former is in I.
Similarly, $y_{i_2}^{q-1} \sum_{\{i_1,i_2\}\subset B} Y_B$ is in I, and by reduction w.r.t. $E_q[Y]$ we have

$$\sum_{\substack{B\in\mathcal{A}\\ \{i_1,i_2\}\subset B}} Y_B \;\in\; I \,. \tag{3}$$

In the same way, restarting from (3), we will eventually deduce that $\sum_{\substack{D\in\mathcal{A}\\ L\subset D}} Y_D$ is in I. Since $|D| = |L|$, then $\sum_{L\subset D} Y_D = Y_L = y_{i_1}\cdots y_{i_l}$, i.e. it is an element of $\mathcal{M}_{m,l}$. But L is a generic element of $\mathcal{A}$ and $\mathcal{M}_{m,l} = \{Y_L\}_{L\in\mathcal{A}}$. □

We now determine the reduced Gröbner basis for $I_{m,t}$.

Theorem 2.4. *Let $t \in \mathbb{N}$ be such that $1 \le t \le m$. Let $G(I_{m,t})$ be the reduced Gröbner basis of $I_{m,t}$. Then:*

$$\begin{aligned} G(I_{m,t}) &= E_q[Y] \cup \mathcal{M}_{m,t}, && \text{for } t \ge 2\,,\\ G(I_{m,t}) &= \{y_1, .., , y_m\}, && \text{for } t = 1\,. \end{aligned}$$

Proof. The statement is easily proved by checking Buchberger's criterion, that is, that all $S-$polynomials coming from $G(I_{m,t})$ are reduced to zero by reduction via $G(I_{m,t})$. □

An obvious consequence of Theorem 2.4 is that $\mathcal{M}_{m,t} \subset I_{m,t}$. For any $1 \le i \le m$, we denote by P_i the set $P_i = \{c \in (\mathbb{F}_q)^m \mid \mathrm{w}(c) = i \}$, and by Q_i the set $Q_i = \sqcup_{0\le j\le i} P_j$. Set P_i contains all vectors of weight i and set Q_i contains all vectors of weight up to i. In the next theorem we describe the Gröbner basis of the vanishing ideal of Q_i.

Theorem 2.5. *Let t be an integer such that $0 \le t \le m-1$. Then*

$$\mathcal{I}(Q_t) \;=\; \langle \{\sigma_{t+1}, \ldots, \sigma_m\} \cup E_q[Y] \rangle \;=\; I_{m,t+1}\,,$$

hence the reduced Gröbner basis G of $\mathcal{I}(Q_t)$ is

$$\begin{aligned} G &= E_q[Y] \cup \mathcal{M}_{m,t}, && \text{for } t \ge 1\,,\\ G &= \{y_1, \ldots, y_m\}, && \text{for } t = 0\,. \end{aligned}$$

Proof. It is enough to show both inclusions

$$\langle \{\sigma_{t+1}, \ldots, \sigma_m\} \cup E_q[Y] \rangle \subseteq \mathcal{I}(Q_t), \quad \mathcal{V}\big(\langle \{\sigma_{t+1}, \ldots, \sigma_m\} \cup E_q[Y] \rangle\big) \subseteq \mathcal{V}(\mathcal{I}(Q_t))\,.$$

For any c in Q_t, we have $c \in (\mathbb{F}_q)^m$ and $\sigma_{t+1}(c) = 0, \ldots, \sigma_m(c) = 0$, hence $\langle \{\sigma_{t+1}, \ldots, \sigma_m\} \cup E_q[Y] \rangle \subseteq \mathcal{I}(Q_t)$.

Since $\mathcal{V}(\mathcal{I}(Q_t)) = Q_t$, we need to show

$$a \in \mathcal{V}(\langle \{\sigma_{t+1}, \ldots, \sigma_m\} \cup E_q[Y] \rangle) \implies a \in Q_t \,.$$

We use the relation $(\mathbb{F}_q)^m = P_0 \sqcup P_1 \sqcup \ldots \sqcup P_m$ and we observe that $\mathcal{V}(\langle \{\sigma_{t+1}, \ldots, \sigma_m\} \cup E_q[Y] \rangle) \neq \emptyset$, since it contains the zero vector.
Suppose by contradiction that there is an $a \in P_{t+1} \sqcup P_{t+2} \sqcup \ldots \sqcup P_m$ such that $\sigma_{t+1}(a) = 0, \ldots, \sigma_m(a) = 0$. Since a has weight r, with $r \geq t+1$, it follows that there exists a monomial $\overline{\mathsf{m}} \in \mathcal{M}_{m,r}$ such that $\overline{\mathsf{m}}(a) \neq 0$ and $\mathsf{m}(a) = 0$ for any other monomial m in $\mathcal{M}_{m,r}$. Hence $\sigma_r(a) \neq 0$, contradicting the hypothesis $a \in \mathcal{V}(\langle \{\sigma_{t+1}, \ldots, \sigma_m\} \cup E_q[Y] \rangle)$. □

3 A distance-computing algorithm

In this section we propose a computational method to find the weight distribution of a code C in $\mathcal{C}(n, k, q)$. We also extend this method to find the distance and the distance distribution of C. From now on, t will be understood to satisfy $1 \leq t \leq n$ and C will denote a code in $\mathcal{C}(n, k, q)$.

3.1 Gröbner basis of a non-linear systematic code

We apply our previous results to give a structure for the Gröbner basis of C. Let $n, k \in \mathbb{N}$ and $\mathbb{F}_q[X, Z]$ be as in Section 2. We also use $E_q[X] \subset \mathbb{F}_q[X, Z]$ with the obvious meaning.

We can view C as a set of points in $(\mathbb{F}_q)^n \subset (\overline{\mathbb{F}}_q)^n$ and hence as a 0-dimensional variety, so that $\mathcal{I}(C)$ is its vanishing ideal in $\mathbb{F}_q[X, Z]$. We describe the reduced Gröbner basis of $\mathcal{I}(C)$ w.r.t. lex.

Theorem 3.1. *Let G be the reduced Gröbner basis for $I = \mathcal{I}(C)$ w.r.t. lex with $x_1 < \cdots < x_k < z_1 < \cdots < z_{n-k}$. Then:*

$$G = E_q[X] \cup \{z_1 - \mathsf{f}_1, \ldots, z_{n-k} - \mathsf{f}_{n-k}\}$$

for some $\mathsf{f}_j \in \mathbb{F}_q[X]$, $1 \leq j \leq n-k$.

Proof. Since $C \subset (\mathbb{F}_q)^n$ and C is systematic, $E_q[X] \subset G$. The existence of polynomials $z_i - \mathsf{f}_i$ follows from the fact that any non-systematic component depends only on the X block of variables. □

Example 3.2. Let C be the following $(4, 2, 2)$ code:

$$C = \{(0,0,0,1), (0,1,0,1), (1,0,0,1), (1,1,0,0)\} \,.$$

The reduced Gröbner basis of $\mathcal{I}(C) \subset \mathbb{F}_2[x_1, x_2, z_1, z_2]$ w.r.t. the lex ordering with $x_1 < x_2 < z_1 < z_2$ is $G(C) = \{x_1^2 + x_1, x_2^2 + x_2, z_1, z_2 + x_1x_2 + 1\}$. So $\mathsf{f}_1 = 0$ and $\mathsf{f}_2 = x_1x_2 + 1$.

Any set of polynomials endowed with the structure of Theorem 3.1 is a Gröbner basis for the ideal generated by itself, and such ideal is zero-dimensional. Hence, the corresponding variety is finite and satisfies the properties of the codes in $\mathcal{C}(n, k, q)$. In this sense we can identify polynomial sets (with the structure of the theorem) and codes in $\mathcal{C}(n, k, q)$. We make it explicit in the following theorem.

Theorem 3.3. *Let $\mathcal{A}_{k,n}$ be the set*

$$\mathcal{A}_{k,n} = \{(\mathsf{f}_1, \ldots, \mathsf{f}_{n-k}) \mid \mathsf{f}_j : (\mathbb{F}_q)^k \longrightarrow \mathbb{F}_q,\ 1 \leq j \leq n-k\}.$$

There is a bijection $\mathcal{A}_{k,n} \leftrightarrow \mathcal{C}(n, k, q)$ given by

$$(\mathsf{f}_1, \ldots, \mathsf{f}_{n-k}) \longleftrightarrow G = E_q[X] \cup \{z_1 - \mathsf{f}_1, \ldots, z_{n-k} - \mathsf{f}_{n-k}\}.$$

3.2 Weight distribution for non-linear systematic codes

The first computational method we propose is a method to obtain the weight distribution for C.

Definition 3.4. *Let $G(C)$ be the reduced Gröbner basis for C, $G(C) = E_q[X] \cup \{z_1 - \mathsf{f}_1(X), \ldots, z_{n-k} - \mathsf{f}_{n-k}(X)\}$. We denote by $\mathcal{W}_C^t$ the following ideal in $\mathbb{F}_q[x_1, \ldots, x_k]$:*

$$\mathcal{W}_C^t = \langle E_q[X] \cup \{\mathsf{m}\big(x_1, \ldots, x_k, \mathsf{f}_1(X), \ldots, \mathsf{f}_{n-k}(X)\big) \mid \mathsf{m} \in \mathcal{M}_{n,t}\}.$$

Lemma 3.5. $\mathcal{V}(\mathcal{W}_C^t) \neq \emptyset \iff \exists c \in C \ \ s.t. \ \ \mathrm{w}(c) \leq t-1.$

Proof. Let $a \in \mathcal{V}(\mathcal{W}_C^t)$, $a = (a_1, \ldots, a_k)$. We have that c is in C, where $c = (a_1, \ldots, a_k, \mathsf{f}_1(a), \ldots, \mathsf{f}_{n-k}(a))$. Since $a \in \mathcal{V}(\mathcal{W}_C^t)$, we have

$$\mathsf{m}(a) = \mathsf{m}\big(a_1, \ldots, a_k, \mathsf{f}_1(a), \ldots, \mathsf{f}_{n-k}(a)\big) = 0 \quad \forall \mathsf{m} \in \mathcal{M}_{n,t}.$$

Which means $c \in \mathcal{V}(I_{m,t})$, hence $c \in Q_{t-1}$ (Theorem 2.5), i.e. $\mathrm{w}(c) \leq t-1$.

The converse can be easily proved by reversing our previous argument. □

Lemma 3.5 shows that a point in $\mathcal{V}(\mathcal{W}_C^t)$ matches a codeword c in C with $\mathrm{w}(c) \leq t-1$, from which we can easily derive the main result of this subsection.

Theorem 3.6. $B_{t-1} = |\mathcal{V}(\mathcal{W}_C^t)| \setminus |\mathcal{V}(\mathcal{W}_C^{t-1})|.$

If C is a distance-invariant code, from the weight distribution of C we can immediately get its distance distribution.

3.3 Distance and distance distribution of non-linear systematic codes

We now propose a computational method to find the distance of C.

Definition 3.7. *Let $\mathsf{f}_1, \ldots, \mathsf{f}_{n-k}$ be as in Theorem 3.1. We denote by $\mathbb{F}_q[X, \tilde{X}]$ the polynomial ring $\mathbb{F}_q[x_1, x_2, \ldots, x_k, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k]$. In the polynomial module $\left(\mathbb{F}_q[X, \tilde{X}]\right)^n$, we denote by $L_{n,k,t}$ the polynomial vector:*

$$L_{n,k,t} = \left(x_1 - \tilde{x}_1, \ldots, x_k - \tilde{x}_k, \mathsf{f}_1(X) - \mathsf{f}_1(\tilde{X}), \ldots, \mathsf{f}_{n-k}(X) - \mathsf{f}_{n-k}(\tilde{X})\right).$$

Definition 3.8. *Let $\mathsf{f}_1, \ldots, \mathsf{f}_{n-k}$ be as in Theorem 3.1. We denote by $\mathcal{I}_C^t$ the ideal in $\mathbb{F}_q[X, \tilde{X}]$ generated by :*

$$\{x_i^q - x_i, \tilde{x_i}^q - \tilde{x_i} \mid 1 \leq i \leq k\} \cup \{\mathsf{m}(L_{n,k,t}) \mid \mathsf{m} \in \mathcal{M}_{n,t}\}.$$

In $(\mathbb{F}_q)^k \times (\mathbb{F}_q)^k$ we denote by Δ_k the diagonal, i.e. the set of points $a = (a_1, \ldots, a_k, \tilde{a_1}, \ldots, \tilde{a_k})$ such that $a_i = \tilde{a}_i$, $1 \leq i \leq k$.

We need to take into account the diagonal because, clearly, $\Delta_k \subset \mathcal{V}(\mathcal{I}_C^t)$ (note that $\Delta_k = \mathcal{V}(\mathcal{I}_C^1)$).

Theorem 3.9. $\mathcal{V}(\mathcal{I}_C^t) \neq \Delta_k \iff \exists\, c_1,\, c_2 \in C$ *such that* $\mathrm{d}(c_1, c_2) \leq t - 1$.

Proof. Let $(a, \tilde{a}) \in (\mathbb{F}_q)^k \times (\mathbb{F}_q)^k \setminus \Delta_k$. We take two codewords c, $\tilde{c}$ such that:

$$c = (a_1, ..., a_k, \mathsf{f}_1(a), \ldots, \mathsf{f}_{n-k}(a)), \quad \tilde{c} = (\tilde{a}_1, \ldots, \tilde{a}_k, \mathsf{f}_1(\tilde{a}), \ldots, \mathsf{f}_{n-k}(\tilde{a})).$$

From this, we obtain that

$$(a, \tilde{a}) \in \mathcal{V}(\mathcal{I}_C^t) \neq \Delta_k \iff \mathsf{m}(c - \tilde{c}) = 0, \quad c \neq \tilde{c},\, \mathsf{m} \in \mathcal{M}_{n,t}\,,$$

since $c - \tilde{c} = (a_1 - \tilde{a}_1, ..., a_k - \tilde{a}_k, \mathsf{f}_1(a) - \mathsf{f}_1(\tilde{a}), ..., \mathsf{f}_{n-k}(a) - \mathsf{f}_{n-k}(\tilde{a}))$. But $\mathsf{m}(c-\tilde{c}) = 0\ \forall \mathsf{m} \in \mathcal{M}_{n,t}$ means $c - \tilde{c} \in \mathcal{V}(I_{n,t})$, which is equivalent to $\mathrm{w}(c-\tilde{c}) \leq t - 1$ (Lemma 3.5). □

We can easily derive the following result.

Corollary 3.10. $\mathcal{V}(\mathcal{I}_C^t) = \Delta_k \iff \mathrm{d}(C) \geq t$.

From Corollary 3.10, an algorithm is directly designed to compute the distance of C.

$j = 2$
While $\mathcal{V}(\mathcal{I}_C^j) = \Delta_k$ **do**
$j := j + 1$;
Output $j - 1$

Example 3.11. Let $C = \{[0,0,2,0], [0,1,0,0], [0,2,0,2], [1,0,2,2], [1,1,2,1], [1,2,1,2], [2,\ 0,\ 1,\ 0], [2,\ 1,\ 1,\ 1], [2,2,0,1]\}$ be a $(4,2,3)$ code. The Gröbner basis $G(C) \in \mathbb{F}_3[x_1, x_2, z_1, z_2]$ w.r.t. lex $x_1 < x_2 < z_1 < z_2$ is

$$G(C) = \{x_1^3 - x_1, x_2^3 - x_2, z_1 - x_2^2 + x_1^2x_2 - x_1^2 + x_1 + 1, z_2 - x_2^2 + x_1x_2 + x_2 - x_1^2 - x_1\}.$$

So $\mathsf{f}_1 = x_2^2 - x_1^2x_2 + x_1^2 - x_1 - 1$ and $\mathsf{f}_2 = x_2^2 - x_1x_2 - x_2 + x_1^2 + x_1$. We choose as monomial order in $\mathbb{F}_3[x_1, x_2, \tilde{x}_1, \tilde{x}_2]$ the degrevlex order with $x_1 > x_2 > \tilde{x}_1 > \tilde{x}_2$. Computing the Gröbner basis of $\mathcal{I}_C^2$, we have $G(\mathcal{I}_C^2) = \{\tilde{x}_1 - x_1, \tilde{x}_2 - x_2, x_1^3 - x_1, x_2^3 - x_2, \tilde{x}_1^3 - x_1, \tilde{x}_2^3 - x_2\}$. Since $\tilde{x}_1 - x_1, \tilde{x}_2 - x_2 \in G$, then $\mathcal{V}(\mathcal{I}_C^2)$ is the diagonal, so $d \geq 2$. Computing then $G(\mathcal{I}_C^3)$, we find

$$G(\mathcal{I}_C^3) = \{\tilde{x}_2^2 - \tilde{x}_2, \tilde{x}_1^2 - \tilde{x}_1, x_2^2 - x_2, x_1^2 - x_1, x_1\tilde{x}_1\tilde{x}_2 - x_1\tilde{x}_2, x_1x_2\tilde{x}_2 - x_2\tilde{x}_1\tilde{x}_2, x_1x_2\tilde{x}_1 - x_2\tilde{x}_1\}\,.$$

This time $\tilde{x}_1 - x_1 \notin G(\mathcal{I}_C^3)$ and the variety $\mathcal{V}(\mathcal{I}_C^3) \neq \Delta_3$, which implies $d = 2$.

Theorem 3.9 suggests a way to compute the distance distribution of C. Indeed, we showed that a point in $\mathcal{V}(\mathcal{I}_C^t)$ is a pair of codewords with distance less than t. From that, we have the following.

Corollary 3.12. $\mathcal{V}(\mathcal{I}_C^t) = \{(c_1, c_2) \mid c_1 \neq c_2 \in C,\ \mathrm{d}(c_1, c_2) \leq t-1\} \cup \Delta_k\,.$

Let $c_1, c_2 \in C$ be such that $\mathrm{d}(c_1, c_2) = i$, for $1 \leq i \leq t-1$. In $\mathcal{V}(\mathcal{I}_C^t)$ there is both the point that matches the pair of words (c_1, c_2) and the point that matches the pair (c_2, c_1). The following fact is then obvious.

Fact 3.13.

$$\sum_{1 \leq i \leq t-1} A_i \;=\; \frac{|\mathcal{V}(\mathcal{I}_C^t)| \setminus |\Delta_k|}{2}\,.$$

Example 3.14. Let $C = \{[0,0,0,0], [0,1,0,0], [0,2,0,0], [1,0,0,0], [1,1,0,2], [1,2,0,2], [2,0,0,2], [2,1,0,0], [2,2,0,0]\}$. Clearly, C is in $\mathcal{C}(4,2,3)$. The distance distribution of C can be determined by hand: $A_1 = 8$, $A_2 = 20$, $A_3 = 8$, $A_4 = 0$. We want to compute all pairs of words (c_1, c_2) with $\mathrm{d}(c_1, c_2) \leq 2$. To accomplish this, we start from the input basis of ideal $\mathcal{I}_C^3$:

$$\begin{aligned}\mathcal{I}_C^3 \;=\; \langle & x_2^2\tilde{x}_1^2\tilde{x}_2 - x_1^2\tilde{x}_1^2\tilde{x}_2 - x_1\tilde{x}_1^2\tilde{x}_2 - \tilde{x}_1^2\tilde{x}_2 - x_1^2x_2^2\tilde{x}_1\tilde{x}_2 + x_2^2\tilde{x}_1\tilde{x}_2 + x_1^2\tilde{x}_1\tilde{x}_2 - \\ & \tilde{x}_1\tilde{x}_2 - x_1^2x_2^2\tilde{x}_2 - x_1^2\tilde{x}_2 + x_1\tilde{x}_2 + x_1^2x_2\tilde{x}_1^2 + x_1x_2\tilde{x}_1^2 - x_1^2x_2 - x_1x_2, x_1^2x_2\tilde{x}_1^2\tilde{x}_2 + \\ & x_1x_2\tilde{x}_1^2\tilde{x}_2 - x_1^2x_2\tilde{x}_2 - x_1x_2\tilde{x}_2 - x_1^2x_2^2\tilde{x}_1^2 - x_1x_2^2\tilde{x}_1^2 + x_1^2x_2^2 + x_1x_2^2,\ \ \tilde{x}_2^3 - \tilde{x}_2,\ \ x_2^3 - \\ & x_2,\ \ \tilde{x}_1^3 - \tilde{x}_1, x_1^3 - x_1, x_2\tilde{x}_1^2\tilde{x}_2^2 + x_2\tilde{x}_1\tilde{x}_2^2 - x_1^2x_2\tilde{x}_2^2 - x_1x_2\tilde{x}_2^2 - x_1^2\tilde{x}_1^2\tilde{x}_2 - x_1\tilde{x}_1^2\tilde{x}_2 - \\ & \tilde{x}_1^2\tilde{x}_2 - x_1^2x_2^2\tilde{x}_1\tilde{x}_2 + x_1^2\tilde{x}_1\tilde{x}_2 - \tilde{x}_1\tilde{x}_2 + x_1x_2^2\tilde{x}_2 - x_1^2\tilde{x}_2 + x_1\tilde{x}_2 + x_1^2x_2\tilde{x}_1^2 + x_1x_2\tilde{x}_1^2 - \\ & x_1^2x_2 - x_1x_2, x_1\tilde{x}_1^2\tilde{x}_2^2 - \tilde{x}_1^2\tilde{x}_2^2 + x_2^2\tilde{x}_1\tilde{x}_2^2 + x_1^2\tilde{x}_1\tilde{x}_2^2 - \tilde{x}_1\tilde{x}_2^2 - x_1x_2^2\tilde{x}_2^2 + x_1^2\tilde{x}_2^2 - x_1\tilde{x}_2^2 - \\ & x_2\tilde{x}_1^2\tilde{x}_2 - x_2\tilde{x}_1\tilde{x}_2 + x_1^2x_2\tilde{x}_2 + x_1x_2\tilde{x}_2 - x_1x_2^2\tilde{x}_1^2 - x_2^2\tilde{x}_1^2 - x_1^2x_2^2\tilde{x}_1 + x_2^2\tilde{x}_1 + x_1^2x_2^2 + \\ & x_1x_2^2, x_1x_2\tilde{x}_1\tilde{x}_2^2 + x_2\tilde{x}_1\tilde{x}_2^2 - x_1^2x_2\tilde{x}_2^2 - x_1x_2\tilde{x}_2^2 - x_1^2\tilde{x}_1^2\tilde{x}_2 + \tilde{x}_1^2\tilde{x}_2 - x_1^2x_2^2\tilde{x}_1\tilde{x}_2 - \\ & x_1x_2^2\tilde{x}_1\tilde{x}_2 - x_1^2\tilde{x}_1\tilde{x}_2 + \tilde{x}_1\tilde{x}_2 + x_1^2x_2^2\tilde{x}_2 + x_1x_2^2\tilde{x}_2 + x_1^2x_2\tilde{x}_1^2 - x_2\tilde{x}_1^2 - x_1^2x_2\tilde{x}_1 + x_2\tilde{x}_1\rangle\end{aligned}$$

From the Gröbner basis we find that $|\mathcal{V}(\mathcal{I}_C^3)| = 65$ ([BCRT93]). Since $|\Delta_2| = 9$, we have $A_1 + A_2 = (65 - 9)/2 = 28$, in accordance with known values.

We can easily get an explicit formula for a generic A_t, since $|\mathcal{V}(\mathcal{I}_C^t)| = 2\sum_1^{t-1} A_i + |\Delta_k|$.

Theorem 3.15.

$$A_t = \frac{|\mathcal{V}(\mathcal{I}_C^{t+1})| \setminus |\mathcal{V}(\mathcal{I}_C^t)|}{2}$$

We provide a last example.

Example 3.16. For the same code of Example 3.14, we find A_2.

$\mathcal{I}_C^2 = \langle \tilde{x}_1\tilde{x}_2 - x_1\tilde{x}_2 - x_2\tilde{x}_1 + x_1x_2, \tilde{x}_2^3 - \tilde{x}_2, x_2^3 - x_2, x_1^3 - x_1, x_1\tilde{x}_2^2 - x_1x_2^2, x_1x_2^2\tilde{x}_2 - x_1\tilde{x}_2, \tilde{x}_1^2 - x_2^2\tilde{x}_1 - \tilde{x}_1 + x_1x_2^2 - x_1^2 + x_1, x_1x_2^2\tilde{x}_1 - x_1^2\tilde{x}_1 + x_1\tilde{x}_1 - x_1^2x_2^2 - x_1^2 + x_1, x_1^2x_2\tilde{x}_1 + x_1x_2\tilde{x}_1 - x_1^2x_2 - x_1x_2 \rangle$

From the Gröbner basis we obtain that $|\mathcal{V}(\mathcal{I}_C^2)| = 25$. Applying Theorem 3.15, we can thus conclude $A_2 = (65 - 25)/2 = 20$.

4 Complexity considerations

The complexity of our method (Corollary 3.10) is basically the complexity of the computation of a Gröbner basis for $\mathcal{I}_C^t$, $2 \leq t \leq d+1$, which is the core of the algorithm. The ideal requiring a bigger computational effort is obviously $\mathcal{I}_C^t$ with the largest t that we have to compute. From now on, we try to consider the ideal $\mathcal{I}_C^t$ where the f polynomials are as generic as possible.

To compute the distance, we do not need the exact structure of the Gröbner basis, but we just want to know if $\mathcal{V}(\mathcal{I}_C^t)$ is the diagonal. We experimentally found that, if $\mathcal{V}(\mathcal{I}_C^t)$ is the diagonal, then the basis obtained by auto-reductions is the diagonal as well. Thus, in our case the complexity of the Gröbner computation seems to be equivalent [1] to that of the computation of an inter-reduced basis from the proposed generator set for $\mathcal{I}_C^t$. Hence, we focus on the complexity estimation for the relevant inter-reductions.

We want to estimate the complexity to get an inter-reduced basis, starting from the input basis of generators for $\mathcal{I}_C^t$. We will use some experimental results. Let L be a set of polynomials that we want to inter-reduce. Let $N = |L|$. A single reduction involves an element f of L that is reduced w.r.t. $L \setminus \{f\}$. The output is either 0, in which case L becomes $L \setminus \{f\}$ (and N becomes $N-1$), or f itself, if it cannot be reduced, or a new polynomial f', which is the remainder and which replaces f in L. So, in the worst case, any reduction requires a division by $N-1$ polynomials and generates a new polynomial f'. Note that the L size cannot increase. The starting step requires then, in the worst case, $\frac{N(N-1)}{2}$ divisions. Let r be the number of new polynomials which are obtained by reductions, both at the starting step and at any other subsequent step. Since N does not grow, any of these r new polynomials has to

[1] They give different results, except when they output the diagonal.

be divided by no more than $N-1$ polynomials. In conclusion, the worst case estimate gives $\frac{N(N-1)}{2} + rN$ divisions, and $2^n n$ is clearly the (worst case) cost of any division.
In our case, $N = \binom{n}{k} = \binom{2k}{k}$. By induction, we can derive the following lemma:

Lemma 4.1. *Let $n, k \in \mathbb{N}$, with $1 \leq k \leq n$, and $\alpha \in \mathbb{Q}$. If $s(\alpha)$ is such that $1 + 2^\alpha \leq 2^{s(\alpha)}$, then:*

$$\binom{n}{k} 2^{\alpha k} \leq 2^{s(\alpha) n}\,.$$

We denote by N the number of polynomials in

$$S_{gen} = \{x_i^q - x_i, \tilde{x_i}^q - \tilde{x_i} \mid 1 \leq i \leq k\} \cup \{\mathsf{m}(L_{n,k,t}) \mid \mathsf{m} \in \mathcal{M}_{n,t}\}\,.$$

We treat the hard case, i.e. when $n = 2k$ and $d \simeq k$. Despite our worst case formula, our computations suggest that it is possible to assume:

- N initial divisions instead of $\frac{N(N-1)}{2}$,
- $r = \sqrt{2^{2k}} = 2^k$ (2^{2k} is the number of all possible monomials),
- $2^k n$ instead of $2^n n$, the average cost of each division,
- $\sqrt{N}$ the average number of divisions for any new polynomial f'.

Thus, the computational cost C to produce an inter-reduced basis from S_{gen} can be estimated in

$$\begin{aligned}\mathsf{C} = \left(\binom{2k}{k} + 2^k\sqrt{\binom{2k}{k}}\right) 2^k n &= \left[\binom{2k}{k}2^k + 2^{\frac{3}{2}k}\sqrt{\binom{2k}{k}2^k}\right] 2k \\ &= \left[2^{3k} + 2^{\frac{3}{2}k}2^{\frac{3}{2}k}\right] 2k = (2^{3k} + 2^{3k})2k = 2^{3k+1}2k,\end{aligned}$$

where we applied Lemma 4.1 with $\alpha = 1$ and $s(\alpha) = 1.5$ [2]. We would like to compare our estimates with our numerical results. In order to do that we highlight the asymptotic exponential behaviour in k.

Definition 4.2. *Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ s.t. $f(m) > 0$, $g(m) > 0$ for $m \geq 1$. We say that $f \simeq g$ if and only if there exist $m_1, m_2, \alpha_1, \alpha_2 \geq 1$ such that*

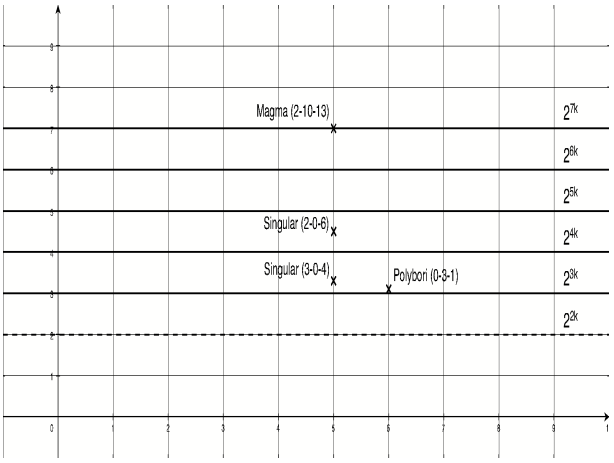
$$f(m) \leq g(m)m^{\alpha_1} \quad m \geq m_1, \qquad g(m) \leq f(m)m^{\alpha_2} \quad m \geq m_2.$$

It is easy to see that $\simeq$ is an equivalence relation and, given $f \simeq 2^{\alpha n}$ and $g \simeq 2^{\beta n}$, for $\alpha, \beta \geq 1$, $\alpha, \beta \in \mathbb{R}$, then $f \simeq g$ if and only if $\alpha = \beta$. Moreover $\alpha > \beta$ implies $\lim_{n\to\infty} \frac{g}{f} = 0$.

With this notation, our previous estimates for C can be written as $\mathsf{C} \simeq 2^{3k}$.

We tested our problem using different computer algebra systems (Magma

[2] actually $s(\alpha)$ is slightly larger.



2.10.13 [MAG], Polybori 0.3.1 [pol], Singular 2.0.6, Singular 3.0.4 [GPS07]). For any system the time needed has a behavior of kind $2^{\alpha k}$, with α depending on the system. In particular, we report the graph where $x = k$ and $y = \log(\frac{time\,in\,k}{time\,in\,k-1})$, so that the y values represent the expected exponent. This suggests us the following values for the computational costs: 2^{7k} for Magma 2.10.13, $2^{4.5k}$ for Singular 2.0.6, $2^{3.5k}$ for Singular 3.0.4, 2^{3k} for Polybori 0.3.1. We note that a brute-force check of the distance has an asymptotic behaviour like 2^{2k}. Unfortunately, it is impossible to find an algorithm [3] that computes the closest pairs of a code with a lower complexity, since 2^{2k} is indeed the complexity of the problem (see Section 4.1). admittedly, 2^{2k} looks better than our estimate 2^{3k}, but two comments are in order:

- If we examine the drastic improvement obtained by the evolution of computer algebra systems (from 2^{7k} to 2^{3k} in few years), we can reasonably assume that our estimates are still pessimistic and that further improvements in software development will allow our method to run like $2^{\alpha k}$, with $2 < \alpha < 3$, possibly close to $\alpha = 2$.
- The brute-force check can output the distance only if the code is given. When a family of codes is given, that is, the $\{\mathrm{f_i}\}$ depend on some parameters $\{\lambda_j\}_{j\in J}$, the check is inapplicable, but our approach may be able to give general results for the family, such as a lower bound on the distance independent of the λ_j's or even a "a priori" bound dependent on the λ_j's.

4.1 Complexity of the closest-pair problems

The determination of the minimum distance of a non-linear code is an instance of the more general "closest pair problem", which is studied for general metric spaces and, more deeply, for the Euclidean spaces $\mathbb{R}^n$. Several related problems are studied in this context, such as the "nearest neighbour problem"

[3] The situation in the linear case is totally different. The problem of finding the distance is NP-complete ([Var97]), so no sub-exponential algorithm is known, but it might exist.

(which is the "decoding problem" in coding theory). An excellent reference is [PS85], to which we implicitly refer when we do not give explicit definitions or quotations.

From now on, let S be the number of points we are considering. In the breakthrough 1975 paper [SH75] the complexity of many of these problems was established for the planar Euclidean case (i.e., for $\mathbb{R}^2$). In particular, it was shown that it is possible to solve the closest-pair problem in $S\log(S)$ steps, by a clever application of the Voronoi diagrams. And it was also shown that $S\log(S)$ matches exactly the complexity of the problem. Since S^2 is the complexity of the "naive approach[4]", one might think that the naive approach could be beaten also in other metric spaces. Later, more refined algorithms have appeared that solve the closest-pair problem over $\mathbb{R}^n$ with a claimed $S\log(S)$ complexity. However, these algorithms' complexity is actually $S\log(S)f(n)$ for some function $f : \mathbb{N} \to \mathbb{N}$, that is, their complexity is computed by considering the space dimension *fixed*. The fact here is that computational geometers are interested in cases when the number of points is much larger than the space dimension. In the Hamming space $(\mathbb{F}_2)^n$ it is crucial to write explicitly also the dependence on n, since clearly $S \leq 2^n$. A deep analysis of their proofs shows that f is exponential in n (e.g., about 4^n in Suri's divide-and-conquer algorithm [Sur09]). Translated into Hamming space language, this means that the naive algorithm performs no worse than the others. In fact, even if we allow for $f = 2^n$, we will get an overall complexity of $S\log(S)2^n = 2^k \cdot k \cdot 2^n \sim 2^{3k}$ (2^{2k} is the complexity of the naive approach). Of course, we would expect that the dependence on the dimension could be improved.

Remark 4.3. The instance in $(\mathbb{F}_2)^n$ of the closest-pair problem is actually different from the "distance computation problem", since the latter needs only to output the value of the distance and not the closest pair. However, both our Gröbner basis algorithm and the naive approach do output the closest pair/s, and so we will consider the former from now on.

When not dealing with Euclidean spaces (or metric spaces embeddable with an isometry into $\mathbb{R}^n$), the standard model in the literature for these problems is the "black box with distance oracle". Basically it means that the complexity is computed in terms of the number of distance calculations which are necessary (a "call to the distance oracle" means a distance calculation). As an example, it is possible to adapt the proof of Theorem 2.3 in [KL05] to prove the following

Theorem 4.4. *The (worst-case) complexity of decoding non-linear codes in* $(\mathbb{F}_2)^n$ *is* $\Omega(S)$.

Before we prove it, we need to explain the meaning of a "proof". We are assuming that there is an algorithm $\mathcal{A}$, which accepts as input both a non-

[4] this is the way the "brute-force check" is called in computational geometry.

linear code X of size S and a query q, i.e. a point for which we must find the point/s in X closest to q. Algorithm $\mathcal{A}$ is the best possible algorithm. It calls the distance oracle many times and tries to use the information on the computed distances to avoid making other distance computations. What $\mathcal{A}$ can use is the triangle inequality:

$$\mathrm{d}(x,y) \leq \mathrm{d}(x,z) + \mathrm{d}(y,z), \quad \mathrm{d}(x,z) \geq \mathrm{d}(x,y) - \mathrm{d}(y,z) \tag{4}$$

since the algorithm is searching a distance minimum, it will use the right-hand version of (4). To prove Theorem 4.4 we need to exhibit, for any n sufficiently large, a code of length n and a query q such that $\mathcal{A}$ is forced to perform $\Omega(S)$ distance computations.

Proof. Let $\ell = \lfloor \frac{n}{2} \rfloor$. Let X_n be the code in $(\mathbb{F}_2)^n$ for $n \geq 4$ with query q_n

$$X_n = \{c \in (\mathbb{F}_2)^n \mid \mathrm{w}(c) = \ell\} \cup \{P = (1,0,\ldots,0)\}, \quad q_n = \{(0,\ldots,0)\}.$$

In other words, q_n is the zero vector and X_n is the sphere centered in q_n of radius ℓ plus a point P at distance 1 from q_n. The crux of the proof here is that $\mathcal{A}$ will always get ℓ from its distance computations, except for $\mathrm{d}(q_n, P)$, which will give 1. So, it does not matter how smart $\mathcal{A}$ is, it will *not able* to use any of its former distance computations. Therefore, in the worst case, $\mathcal{A}$ has to try all $\mathrm{d}(q_n, x)$ for $x \in X$. □

Note that the size of X_n in the above proof grows with n (actually $|X_n| > 2^{n/2}$). Without this property, we could take X_n with two points and then of course any algorithm will need to perform $1 = S - 1 = \Omega(S)$ distance computations.

The previous digression is important in our opinion to put into context the problem and understand the proof of our last result in this section, which is the following theorem.

Theorem 4.5. *The (worst-case) complexity of computing the closest codeword pairs of a binary non-linear code is* $\Omega(S^2)$.

Proof. For any subset Y of $(\mathbb{F}_2)^n$ we denote by D its diameter, that is, the maximum distance between two points in Y, and by d its (Hamming) distance. Let $\mathcal{B}$ be any algorithm having as input a non-linear code and returning a pair of closest codewords (we can think of $\mathcal{B}$ as the best possible). To prove our claim we need to show that for any sufficiently large n we can find a code X_n such that $\mathcal{B}$ cannot use any distance computations already performed in order to discard other distance computations.

We consider $n \geq 10$. We take X_n as any maximal subset of $(\mathbb{F}_2)^n$ such that its *aspect ratio* $\frac{D}{d}$ is strictly lower than 2. The bound obtainable by $\mathcal{B}$ from any of its former computations is at most

$$\mathrm{d}(x,z) \geq \max \mathrm{d}(x,y) - \min \mathrm{d}(y,z) = D - d < 2d - d = d\,.$$

This cannot give any help to $\mathcal{B}$ and so $\mathcal{B}$ is forced to compute also $\mathrm{d}(x,z)$. □

We would like to give one final remark. The size of code X_n in the previous proof clearly grows with n, but it will have only a few words. If one is interested in building a larger code on which $\mathcal{B}$ needs to compute $\Omega(S^2)$, then one will need to use Ramsey-like properties of the Hamming space ([BLMN05]), similarly to what is done in [KL05]. To say it in a few words avoiding technicalities, it is possible to find a subset Y of $(\mathbb{F}_2)^n$ with a size $2^{\Omega(n)}$ and such that its aspect ratio is limited to $2+\epsilon$, for some small ϵ. Since the number of minimum-distance word pairs and the number of maximum-distance word pairs may be "small" compared to the number of all word pairs, it follows that $\mathcal{B}$ may have to examine in the worst case a number $\Omega(S^2)$ of word pairs before being able to use (4). However, we feel that a complete proof for this claim is out of the scope of this paper and so we do not delve into it.

5 Conclusions

The decoding performance of a distance-invariant code (e.g., a linear code) depends on its weight distribution, although already the distance gives partial information on it. For a generic non-linear code the performance depends on its distance distribution (but the distance provides significant information).
In this paper we have provided some Gröbner techniques in order to compute the above-mentioned code parameters in the systematic case (which is the most interesting). We realize that no method can be faster than running a specific C-programme optimized for a given code and we are far from claiming that our techniques can compete with this approach. However, the optimized software has two drawbacks:

- the software writing and debugging can be long and complex, (while our methods are very easy to implement using a software package for Gröbner basis computations),
- the software programme can be used only on a given code and cannot give general results on a code family (while our methods could).

In our opinion this means that our methods can be of interest for a mathematician investigating theoretical code properties.

Acknowledgements

Part of these results can be found in [Gue05] and [Gue09] and have been presented at MEGA2005 and Linz D1 2006 ([GOS09]), which was a workshop within the Special Semester on Groebner Bases, February–July 2006, organized by RICAM, Austrian Academy of Sciences, and RISC, Johannes Kepler University, Linz, Austria.

The first two authors would like to thank their supervisor: the third author.

For their comments and suggestions, the authors heartily thank the anonymous referees and the following people: F. Caruso, P. Fitzpatrick, P. Gianni, R. Krauthgamer, T. Mora, I. Simonetti and C. Traverso.

The authors would also like to thank the team at the computational centre MEDICIS (http://www.medicis.polytechnique.fr/).

This work has been partially supported by STMicroelectronics contract "Complexity issues in algebraic Coding Theory and Cryptography".

References

[BCRT93] A. M. Bigatti, P. Conti, L. Robbiano, and C. Traverso, *A "divide and conquer" algorithm for Hilbert-Poincaré series, multiplicity and dimension of monomial ideals*, Applied algebra, algebraic algorithms and error-correcting codes, LNCS, vol. 673, Springer, Berlin, 1993, pp. 76–88.

[BLMN05] Y. Bartal, N. Linial, M. Mendel, and A. Naor, *On metric Ramsey-type phenomena*, Ann. of Math. (2) **162** (2005), no. 2, 643–709.

[Buc65] Bruno Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*, Ph.D. thesis, Innsbruck, 1965.

[Buc06] B. Buchberger, *Bruno Buchberger's PhD thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal*, J. Symb. Comput. **41** (2006), no. 3-4, 475–511.

[BvLW83] R. D. Baker, J. H. van Lint, and R. M. Wilson, *On the Preparata and Goethals codes*, IEEE Trans. on Inf. Th. **29** (1983), no. 3, 342–345.

[CLO92] D. Cox, J. Little, and D. O'Shea, *Ideals, varieties, and algorithms*, Springer-Verlag, 1992, An introduction to computational algebraic geometry and commutative algebra.

[Gal63] R. Gallager, *Low-density parity-check codes*, Ph.D. thesis, Massachusetts Institute of Technology, 1963.

[GOS09] E. Guerrini, E. Orsini, and I. Simonetti, *Gröbner bases for the distance distribution of systematic codes*, Gröbner Bases, Coding, and Cryptography (M. Sala, T. Mora, L. Perret, S. Sakata, and C. Traverso, eds.), RISC Book Series, Springer, Heidelberg, 2009, pp. 367–372.

[GPS07] G.-M. Greuel, G. Pfister, and H. Schönemann, *Singular 3.0. A computer algebra system for polynomial computations*, http://www.singular.uni-kl.de, 2007, Centre for Computer Algebra, University of Kaiserslautern.

[Gue05] Eleonora Guerrini, *On distance and optimality in non-linear codes*, Master's thesis (laurea), Univ. of Pisa, Dept. of Math., 2005.

[Gue09] ———, *Systematic codes and polynomial ideals*, Ph.D. thesis, University of Trento, 2009.

[HKC+94] A. R. Hammons, Jr., P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, and P. Solé, *The $\mathbf{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes*, IEEE Trans. on Inf. Th. **40** (1994), no. 2, 301–319.

[KL05] R. Krauthgamer and J. R. Lee, *The black-box complexity of nearest-neighbor search*, Theoret. Comput. Sci. **348** (2005), no. 2-3, 262–276.

[MAG] *MAGMA: Computational Algebra System for Algebra, Number Theory and Geometry*, The University of Sydney Computational Algebra Group., http://magma.maths.usyd.edu.au/magma.

[PHB98] V. S. Pless, W. C. Huffman, and R. A. Brualdi (eds.), *Handbook of Coding Theory. Vol. I, II*, North-Holland, Amsterdam, 1998.

[pol] *The software package PolyBori - Polynomials over Boolean Rings*, http://polybori.sourceforge.net/.

[Pre68] F. P. Preparata, *A class of optimum nonlinear double-error correcting codes*, Inform. Control **13** (1968), no. 13, 378–400.

[PS85] F. P. Preparata and M. I. Shamos, *Computational geometry*, Texts and Monographs in Computer Science, Springer, 1985, An introduction.

[SH75] M. I. Shamos and D. H., *Closest-point problems*, Proc. of Annual Symposium on Foundations of Computer Science (1975), IEEE Computer Society, 1975, pp. 151–162.

[Sha48] C. E. Shannon, *A mathematical theory of communication*, Bell System Tech. J. **27** (1948), 379–423, 623–656.

[Sur09] S. Suri, *Closest pair problem*, Tech. report, Dept. of CS, Univ. of California, 2009, http://www.cs.ucsb.edu/ suri/cs235/ClosestPair.pdf.

[Var97] A. Vardy, *The intractability of computing the minimum distance of a code*, IEEE Trans. on Inf. Th. **43** (1997), no. 6, 1757–1766.